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## A Comprehensive Literature Review of Fuzzy Differential Equations with Applications

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### Abstract

Physical dynamical systems can be described through mathematical models using the theory of differential equations. Many different types of uncertain healing occur in the real-world. Fuzzy logic is an effective mathematical tool for defining the sense of non-random uncertainty. Since fuzzy arithmetic differs from its counterparts, it would be more reliable if we use the differential equation to construct models involving uncertain decision parameters. In this paper, we attempt a detailed review of the existing literature related to the theory and application of fuzzy differential equations. The scientific review of this paper includes surveys of the Literature involving different definitions of fuzzy derivation. Distinguished solution approaches and applications of fuzzy differential equations in the fields of science, technology, and management are discussed in this paper. We also provide hints regarding future research challenges and scopes with the theory of fuzzy differential equations. This paper may be impactful documentation of the history of differential equations linking the past, present and future of the concerned research topic.

**Keywords:** Fuzzy differential equations, Fuzzy derivatives, Fuzzy sets, Fuzzy numbers.

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## 1|Introduction

### Basic Idea on Fuzzy Set - Fuzzy Number - Application

*Notion of Fuzzy Set Theory:* Problems in the real world frequently become complicated due to an element of uncertainty in the parameters that characterize the problem or in the environment in which the problem occurs, given the growing complexity of the system. However, statistical interference becomes extremely challenging when dealing with systems that have limited data accessible. Although the probability technique has been effectively used to solve many engineering problems, the probabilistic method still has certain drawbacks. For instance, to reach the necessary confidence level, probabilistic approaches depend on the mass collection of random data. However, in complex large-scale systems, there is a great deal of non-random uncertainty, making it challenging to determine the precise probability of an event. Logic and fuzzy set theory-based approaches, in addition to probability theory, offer a helpful tool for handling uncertainty and helping to overcome these challenges. Fuzzy set theory is a useful tool which was first developed by Zadeh [1].

However, Heilpern [2] presented the idea of fuzzy mapping, which is the mapping of an arbitrary set in a metric linear space  $X$  to one subgroup of fuzzy sets. The fixed-point theorem for fuzzy mappings was also demonstrated by him [3].

Subsequently, Atanassov [4] created the intuitionistic fuzzy set idea, in which membership and non-membership values are present in every set element. Diamond and Kloeden [5] provided a characterization of compact subsets for the metric space of convex fuzzy, normal fuzzy sets on the base space  $R^n$ .

After that, Diamond [6] defined fuzzy sets with fuzzy star shapes. Zhang [7] introduced an extension of fuzzy set namely bipolar fuzzy set whose membership value lies within  $[-1, 1]$ . In a bipolar fuzzy set, an element with membership degree 0 signifies its lack of relevance to the corresponding property; an element with membership degree  $(0, 1]$  suggests that it partially satisfies the property; and an element with membership degree  $[-1, 0]$  suggests that it partially satisfies the implicit counter-property. Piegat [8] introduced a new definition of fuzzy set where the fuzziness decreases according to number of arithmetic operations. However, Torra and Narukawa [9] defined another extension of fuzzy set namely hesitant fuzzy set. Pythagorean fuzzy subsets are a novel type of non-standard fuzzy subsets that Yager [10] presented, along with the associated concept of Pythagorean membership grades.

He also addressed the negation operation and how the Pythagorean theorem relates to it. De and Beg [11] used the dense fuzzy concept in the Neutrosophic fuzzy set. After that, De [12] investigated a triangular dense fuzzy lock set on a Cauchy sequence that possessed a unique characteristic. In this set, normality won't ever be reached unless the triangular dense fuzzy set is unlocked with a unique key in its ultimate defuzzified condition. However, the conventional ideas regarding the fuzziness of the fuzzy parameters are always the same, but in reality, over time, the system's fuzziness started to disappear because of human experience and accuracy. In light of this, De and Mahata [13] defined a cloudy fuzzy set, the components of which are functions of time, and which transform into a crisp set after an unlimited amount of time. Cuong [14] introduced the concept of a picture fuzzy set, which is a further development of an intuitionistic fuzzy set. Chen et al. [15] defined m-Polar fuzzy sets which is an extension of bipolar fuzzy sets.

*Introduction to Fuzzy Number:* Bellman and Zadeh [16] used the novel idea to solve decision-making problems of operations research. Several operations on fuzzy numbers were specified for the first time by Dubois and Prade [17]. A fuzzy number is an extension of a conventional real number in that it refers to a connected set of possible values rather than a single one, with each value having a weight ranging from 0 to 1. Consequently, a convex normalized fuzzy set of the real line is a specific instance of a fuzzy number. The reduction of a fuzzy number to a real number was expressed by Bobylev [18]. After that, Chutia et al. [19] developed an alternative method of finding the membership of a fuzzy number. Mukherjee, A. K. et al. [20] develop arithmetic operations of fuzzy numbers using  $\alpha$ -cut approach. There exist different types of fuzzy numbers, such as: Triangular Fuzzy Number [21, 22, 23], Trapezoidal Fuzzy Number [24, 25, 26], Pentagonal Fuzzy Number [27, 28, 29], Hexagonal

Fuzzy Number [30, 31, 32], Interval valued Fuzzy Number [33, 34, 35], Bell Shaped Fuzzy Number,[36, 37, 38], Time Dependent Fuzzy Number [39, 40, 41, 42], Non-Linear Fuzzy Number [22, 43, 44], Gaussian Fuzzy Number [45, 46, 47] and so on.

*Concept about Fuzzy Function:* De and Beg [48] first defined the dense fuzzy set whose components are considered as a sequence of functions. The concept of a fuzzy function is primarily recognized in the literature as a distinct type of fuzzy relation, one that "acknowledges" two specified similarities within the universe of discourse. In this context, a fuzzy function appears akin to a blurred crisp function. Dickerson and Kosko [49] demonstrated the approximation of a function using an additive fuzzy system, which involves covering the function's graph with ellipsoidal rule patches. After that, Perfilieva [50] introduced the concept of a fuzzy function and its representation through a fuzzy relation. Fuzzy functions can be employed as components in fuzzy relation equations, providing a way to model and represent complex relationships between fuzzy sets. Demirci [51] established the introduction of several kinds of fuzzy functions together with their basic characteristics. Demirci [52] claimed some application of fuzzy functions. However, for the creation of fuzzy system models, Turksen [53] suggested the least squares estimation (LSE) technique as the method for determining "Fuzzy Functions". Moreover, for fuzzy valued functions, Bede and Stefanini [54] presented and examined new generalized differentiability notions using creative expansions of the Hukuhara difference for fuzzy sets. Beyhan and Alci [55] used the fuzzy function concept to improve auto regressive with exogenous input modelling. A differentiability of the type-2 fuzzy number-valued functions was defined by Mazandarani and Najariyan [56]. Mazandarani et al. [57] also provided a concept of continuous fuzzy functions and a new definition of a granular metric on the space of type-1 fuzzy numbers. Recently, Tripathi [58] developed the idea of the granular F-transform and uses the theory of fuzzy numbers and horizontal membership functions to examine some of its fundamental characteristics.

*Application of Fuzzy Set Theory:* A few applications of fuzzy arithmetic theory were presented by Kaufmann and Gupta [59]. Fuzzy set theory can be applied to almost every branch of mathematics. In a supplier-retailer-customer model operating under a volumetric fuzzy system, De et al. [60] investigated carbon emission issues related to production manufacturing systems in the context of cooperative inventory control and sustainable trade credit financing for degrading items. After that, Maity et al. [61] used the cloudy fuzzy concept to solve an EOQ model of imperfect items with uncertain demand rates. Recently, a green inventory model was examined by Maity et al. [62], in which the selling price, stock level, and green concern level determine the demand rate. They solved the model in a pentagonal intuitionistic dense fuzzy environment. Different researchers use the fuzzy set concept in graph theory also. Singh, P. et al. [63] use a triangular fuzzy set to solve the Fuzzy Mellin Transformation. Bhattacharya and Pal [64] solved the CCTV installation problem using vertex covering of fuzzy graphs. Assessing the likelihood of diverse cancer types affecting various organs within the human body is a customary aspect of decision-making in the field of medicine and healthcare. Recently, Bhattacharya and Pal [65] used fuzzy graphical covers of fuzzy graphs to predict the nature of cancer. However, using the Fuzzy Analytic Hierarchy Process, Paul et al. [66] evaluated five open-ended, large-cap, direct mutual funds with stopped sales. They then used this process to shortlist the most important characteristics relative to a list of criteria. For "multi-period decision-making" problems, Riaz et al. [67] presented several types of dynamic spherical fuzzy aggregation operators. Through multistage, spherical fuzzy dynamic decision analysis, they were able to tackle difficulties related to effective municipal supply chain management. Fahmi et al. [68] used the triangular cubic fuzzy number in multi-criteria decision-making problems. Momena, A. F. et al. [69] applied MCDM based optimization method to choose an Edge computing model in an educational institute and Adhikari, D. et al. [70] utilized entropy-VIKOR based MCDM methodology for ranking different states of India depending on women empowerment.

## Context of Fuzzy Differential Equation

In Mathematics, the primary concept of fuzzy differential equations comes from ordinary differential equations. The differential equation with fuzzy parameters and fuzzy initial values, which formed a sequence of fuzzy numbers is called a Fuzzy Differential Equation [71]. It is also defined as the differential inclusion in a fuzzy set with a non-uniform upper hemicontinuity convex set and compactness.

Fuzzy sets, which are a generalisation of conventional set theory, allow for the representation of uncertainty and ambiguity. The first inventor of this concept is Lotfi A. Zadeh (1965) [72]. The true value of fuzzy logic can be any value between zero to one scale. The particular term of the fuzzy differential equation was formulated by Kandal and Byatt (1978) [73]. The notion of fuzzy derivative idea given by Chang and Zadeh (1972) [74] for 1st time. With the help of the extension principle, Dubois and Prade (1982) [75] defined derivative. Puri and Ralescu [76] presented the H-derivative of the fuzzy valued function of the Hukuhara difference [77]. Kaleva [78] and Seikkala [79] solved togetherly the fuzzy initial value problem with a fuzzy initial condition. The theory applications of fuzzy dynamic systems were also conducted by Kandel and Byatt [80]. The method Hukuhara derivative came from H-difference and Modified Hukuhara derivative was introduced by S. Salahshour, T. Allahviranloo [81]. Mahata, A. et al. [82] use the fuzzy differential equation inclusion extension method to solve the epidemic model.

## Motivation of this study

Differential equations can address the real world's physical processes within their mathematical representatives. Differential equation includes differential coefficient which represents changes of dependent variables with respect to the independent variables. With different perspectives, the differential equation may be of integer and fractional order. Also, there are partial differential equations that impact applied insights. On the other hand, the mathematical modelling of physical processes or the corporate decision phenomena can not be deterministic in the true sense. Instead, such mathematical modelling includes uncertainties regarding involved data and possible outcomes. In this context, mathematical modelling under uncertainty becomes contemporary. Fuzzy set theory has emerged as one of the much-celebrated mathematical tools for dealing with uncertain environments in applied sciences for the last few decades. So, the study of dynamic processes under uncertainties necessitates the simultaneous theory of differential equations and fuzziness. In this context, the theory of fuzzy differential equations came into the literature and became one of the popular mathematical encounters for addressing the modelling and optimization of physical scenarios. Despite of many advanced insights, the theory of fuzzy differential equations has some constraints. Till date, there is no convincing approach to deal with non-linear differential equations under fuzzy uncertainty. However, most of the differential equations associated with the real-world based mathematical models are of non-linear nature. Many existing literature bypassed the issue taking the de-fuzzified information to use in the crisp differential equation. The major drawback of the mentioned approach is that it neglects the calculus of fuzzy valued function having distinction with the crisp analogue. So, there are reasons to be bothered with the study of fuzzy differential equations. Several works have been done with numerous perspectives. However, there are lots of drawbacks to the existing literature. The merits and demerits of fuzzy differential equations motivate us to make a detailed review of fuzzy differential equations, a contemporary topic in literature.

In this review paper, we try to focus on the published fuzzy differential equations applications paper and try to show the applicability of fuzzy differential equations in broader fields. For enthusiastic researchers on the topic of fuzzy differential equations, this article may be regarded as basic documentation and a summary of the existing literature.

## Structure of this paper

In Section 1, we try to focus the basic introductory idea on fuzzy set, fuzzy number and fuzzy differential equations. In Section 2, we addressed basic preliminaries fuzzy numbers, fuzzy functions and fuzzy derivatives. Section 3 is associated with the information of different types of Fuzzy differential equations. The solution procedure for different types of fuzzy differential equations is discussed in Section 4. The main discussion in the topic application of fuzzy differential equations is shown in Section 5. All the fuzzy differential equations applications articles are cited and the main contributions are addressed. Future Research outlines are addressed in Section 6. Lastly, the conclusion of the review paper is discussed in Section 7.

## 2|Preliminaries of Fuzzy Numbers, Fuzzy Functions, and Fuzzy Derivatives

This section discusses fuzzy sets, fuzzy numbers, various properties of fuzzy numbers, fuzzy functions and fuzzy derivatives in detail.

### Fuzzy Set

In 1965, an American mathematician and computer scientist Lotfi A. Zadeh [1] introduced the fuzzy set concept. In the fuzzy set [83], there are ordered pair set elements instead of single elements, first is the element itself followed by its membership degree or degree of belongings. The definition and properties are described as follows:

#### Definition 1. Fuzzy Set

Consider  $U$  be a universal set and  $\tilde{F}$  as a fuzzy set defined on it. The fuzzy set  $(\tilde{F})$  defined as

$$\tilde{F} = \{(x, \mu_{\tilde{F}}) : x \in U\}, \quad (1)$$

where  $\mu_{\tilde{F}} : U \rightarrow [0, 1]$  be a membership function and  $x \in U$  be arbitrary element.

### $\alpha$ -cut of Fuzzy Set

The  $\alpha$ -cut of fuzzy set is a parametric representation of the fuzzy set where it is represented as a classical set. In  $\alpha$ -cut of fuzzy set includes those elements whose membership values are greater than  $\alpha$ . The definitions are as follows:

#### Definition 2. $\alpha$ -cut

Let  $\tilde{H}$  be a fuzzy set with membership function  $\mu_{\tilde{H}}$  defined on a universal set  $U$ . Then

$$\mu_{\alpha} = \{x : \mu_{\tilde{H}}(x) \leq \alpha \& x \in U\}, \quad (2)$$

is called the  $\alpha$ -cut of the fuzzy set  $\tilde{H}$  where  $\alpha \in [0, 1]$ .

#### Definition 3. Strong $\alpha$ -cut

Let  $\tilde{H}$  be a fuzzy set with membership function  $\mu_{\tilde{H}}$  defined on universal set  $U$ . Then

$$\mu_{\alpha} = \{x : \mu_{\tilde{H}}(x) > \alpha \& x \in U\}, \quad (3)$$

is called the strong  $\alpha$ -cut of the fuzzy set  $\tilde{H}$  where  $\alpha \in [0, 1]$ .

### Fuzzy Number

A fuzzy number [84] is a generalisation of a regular real number in that it refers to a connected collection of alternative values rather than a single one, and each possible value has a weight ranging from 0 – 1. The membership function is the name given to this weight. On the other hand, a convex, normalised fuzzy set of the real line is a particular instance of a fuzzy number.

#### Definition 4. Fuzzy Number

A fuzzy set  $\tilde{u} = \{\tilde{u} : \mathbb{R} \rightarrow [0, 1]\}$  is said to be a fuzzy number if it satisfies the following properties,

- (i)  $\tilde{u}$  is normal; i.e., membership value should be 1 for at least one point.
- (ii) The set  $\tilde{u}$  is a convex fuzzy set; i.e.,  $\mu_{\tilde{u}}(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu_{\tilde{u}}(x_1), \mu_{\tilde{u}}(x_2)\}; \forall x_1, x_2 \in \tilde{u}$  and  $0 \leq \lambda \leq 1$ .
- (iii)  $0$ -cut of a fuzzy set  ${}^0\tilde{u} = \{x \in X; \mu_{\tilde{u}}(x) \geq 0\}$  should be bounded.
- (iv) The Membership function of the fuzzy set must be pairwise continuous.

## Fuzzy Number in Parametric form

A collection of equations parameterized with regard to a new parameter is called a fuzzy parametric form and every equation has a different form.

A fuzzy number in parametric form of fuzzy number  $\tilde{u}$  is given as with the help of  $\alpha$ -cut,

$${}^{\alpha}\tilde{u} = [\underline{u}, \bar{u}], \quad (4)$$

satisfies the following conditions

- (i)  $\underline{u}$  is bounded left continuous increasing function in  $[0, 1]$ ,
- (ii)  $\bar{u}$  is bounded right continuous decreasing function in  $[0, 1]$ ,
- (iii)  $\underline{u} \leq \bar{u}$ .

## Fuzzy Arithmetic Operation

Let  $\tilde{u}, \tilde{v} \in \Omega$  are fuzzy numbers and is scalar then the arithmetic operations [85] between and are as follows,

(i) **Addition:**

$${}^{\alpha}\tilde{u} \oplus {}^{\alpha}\tilde{v} = [\underline{u}, \bar{u}] \oplus [\underline{v}, \bar{v}] = [\underline{u} + \underline{v}, \bar{u} + \bar{v}]. \quad (5)$$

(ii) **Scalar Multiplication:**

(a) When  $k > 0$ , then

$$k \otimes {}^{\alpha}\tilde{u} = k[\underline{u}, \bar{u}] = [k\underline{u}, k\bar{u}] ; \forall \alpha \in (0, 1]. \quad (6)$$

(b) When  $k \leq 0$ , then

$$k \otimes {}^{\alpha}\tilde{u} = k[\underline{u}, \bar{u}] = [k\bar{u}, k\underline{u}] ; \forall \alpha \in (0, 1]. \quad (7)$$

(iii) **Multiplication:**

$${}^{\alpha}\tilde{u} \otimes {}^{\alpha}\tilde{v} = [\underline{u}, \bar{u}] \otimes [\underline{v}, \bar{v}] = [\underline{w}, \bar{w}], \quad (8)$$

where  $\underline{w} = \min\{\underline{u}\underline{v}, \underline{u}\bar{v}, \bar{u}\underline{v}, \bar{u}\bar{v}\}$  and  $\bar{w} = \max\{\underline{u}\underline{v}, \underline{u}\bar{v}, \bar{u}\underline{v}, \bar{u}\bar{v}\}$ .

## Different Type of Fuzzy Derivatives

*Hukuhara Derivative:* Hukuhara Derivative, which defined the “H-difference” to overcome the problem of finding a suitable difference between two intervals for interval-valued functions. So, first, we define the Hukuhara difference and with the help of examples, we observe for what conditions it exists or not.

(i) The parametric forms of the H-difference between two compact intervals  ${}^{\alpha}\tilde{x} = [\underline{x}, \bar{x}]$  and  ${}^{\alpha}\tilde{y} = [\underline{y}, \bar{y}]$  is defined as,

$${}^{\alpha}\tilde{x} \ominus {}^{\alpha}\tilde{y} = [\underline{x} - \underline{y}, \bar{x} - \bar{y}]. \quad (9)$$

(ii) Let  $\tilde{x}, \tilde{y} \in E$  if there exists  $\tilde{z} \in E$  such that  $\tilde{x} = \tilde{y} \oplus \tilde{z}$  then  $\tilde{z}$  is called the H-difference of  $\tilde{x}$  and  $\tilde{y}$  and it is denoted by

$$\tilde{x} \ominus_H \tilde{y} \cdot \tilde{x} \ominus_H \tilde{y} \neq \tilde{x} + (-1)\tilde{y}. \quad (10)$$

Thus, the Hukuhara derivative is defined as,

Consider, a fuzzy mapping  $\tilde{f} : I \rightarrow E$  and  $t_0 \in I$  then  $\tilde{f}$  is said to be differentiable at  $t_0$ , as in, if there exists an element  $\dot{\tilde{f}}(t_0)$  such that for all small and the limits,

$$\lim_{h \rightarrow 0^+} \frac{\tilde{f}(t_0 + h) \ominus \tilde{f}(t_0)}{h} = \lim_{h \rightarrow 0^-} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)}{h} = \dot{\tilde{f}}(t_0). \quad (11)$$

The following examples explain how to find the Hukuhara difference.

**Example 1.** When the length of the interval of a fuzzy number is greater than the length of the interval of another fuzzy number  $\tilde{v}$ :

Let  $\tilde{2} = (1, 2, 3)$  and  $\tilde{3} = (2, 3, 4)$  then

$${}^\alpha\tilde{2} = [1 + \alpha, 3 - \alpha] \text{ and } {}^\alpha\tilde{3} = [2 + \alpha, 4 - \alpha].$$

To, visualize fuzzy difference of  $\tilde{2} \ominus_H \tilde{3}$ , first we convert it into the crisp form using  $\alpha$ -cut,

$$\begin{aligned} {}^\alpha\tilde{3} \ominus_H {}^\alpha\tilde{2} &= [2 + \alpha, 4 - \alpha] \ominus_H [1 + \alpha, 3 - \alpha] \\ &= [2 + \alpha - 1 - \alpha, 4 - \alpha - 3 + \alpha] \\ &= [1, 1] \\ &= \{1\}. \end{aligned}$$

This example shows that a Hukuhara difference exists for a given condition.

**Example 2.** When a length of interval of a fuzzy number  $\tilde{u}$  is smaller than length of interval of another fuzzy number  $\tilde{v}$ :

$$\begin{aligned} {}^\alpha\tilde{2.5} \ominus_H {}^\alpha\tilde{6} &= [2 + 0.5\alpha, 3 - 0.5\alpha] \ominus_H [5 + \alpha, 7 - \alpha] \\ &= [2 + 0.5\alpha - 5 - \alpha, 3 - 0.5\alpha - 7 + \alpha] \\ &= [-3 - 0.5\alpha, -4 - 0.5\alpha]. \end{aligned}$$

Hukuhara difference does not exist since the definition of the interval is not satisfied in the above Example 2. From Example 2, it is observed that the Hukuhara derivative (based on the Hukuhara difference) does not exist for all cases.

**Example 3.** Find the fuzzy derivative (in the sense of Hukuhara differentiability) of  $\tilde{f}(x) = \tilde{x}^2$  at point  $\tilde{x}_0$ .

**Solution:**

Using the definition of the Hukuhara derivative, we have at  $\tilde{x}_0$ ,

$$\lim_{h \rightarrow 0^+} \frac{\tilde{f}(\tilde{x}_0 + h) \ominus \tilde{f}(\tilde{x}_0)}{h} = \lim_{h \rightarrow 0^-} \frac{\tilde{f}(\tilde{x}_0) \ominus \tilde{f}(\tilde{x}_0 - h)}{h} = \dot{\tilde{f}}(\tilde{x}_0). \quad (12)$$

Let,

$$\lim_{h \rightarrow 0^+} \frac{\tilde{f}(\tilde{x}_0 + h) \ominus \tilde{f}(\tilde{x}_0)}{h} = \lim_{h \rightarrow 0} \frac{\tilde{f}(\tilde{x}_0 + h)^2 \ominus \tilde{x}_0^2}{h}.$$

After applying  $\tilde{\alpha}$ -cut, we have,

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{\tilde{f}(\tilde{x}_0 + h) \ominus \tilde{f}(\tilde{x}_0)}{h} &= \lim_{h \rightarrow 0} \frac{([x_0 + h, \bar{x}_0 + h])^2 - [x_0, \bar{x}_0]^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(x_0 + h)^2, (\bar{x}_0 + h)^2] - [x_0^2, \bar{x}_0^2]}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(x_0 + h)^2 - x_0^2, (\bar{x}_0 + h)^2 - \bar{x}_0^2]}{h} \\ &= \lim_{h \rightarrow 0} \frac{[2x_0h, 2\bar{x}_0h]}{h} \\ &= [2x_0, 2\bar{x}_0]. \end{aligned} \quad (13)$$

Similarly, we can show for,  $\lim_{h \rightarrow 0^-} \frac{\tilde{f}(\tilde{x}_0) \ominus \tilde{f}(\tilde{x}_0 - h)}{h} = [2x_0, 2\bar{x}_0]$ .

The derivative of  $\tilde{f}(x) = x^2$  at point  $\tilde{x}_0$  in parametric form is  $[2x_0, 2\bar{x}_0]$ .

**Note 1.** If  $\tilde{x}_0 > 0$ , then Hukuhara Derivative exist, if we consider  $-1 * \tilde{x}_0$ , then Hukuhara Derivative do not exist.

*Strongly Generalized Differentiability:* As in, consider a fuzzy mapping  $\tilde{f} : I \rightarrow E$  and  $t_0 \in I$  then  $\dot{\tilde{f}}$  is said to be strongly generalized differentiable at  $t_0 \in E$ , if

I. For all  $h > 0$ , sufficiently small  $\exists \tilde{f}(t_0 + h) \ominus \tilde{f}(t_0), \tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)$  exist and the limits,

$$\lim_{h \rightarrow 0^+} \frac{\tilde{f}(t_0 + h) \ominus \tilde{f}(t_0)}{h} = \lim_{h \rightarrow 0^+} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)}{h} = \dot{\tilde{f}}(t_0). \quad (14)$$

II. For all  $h > 0$ , sufficiently small  $\exists \tilde{f}(t_0) \ominus \tilde{f}(t_0 + h), \tilde{f}(t_0 - h) \ominus \tilde{f}(t_0)$  exist and the limits,

$$\lim_{h \rightarrow 0^+} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 + h)}{h} = \lim_{h \rightarrow 0^-} \frac{\tilde{f}(t_0 - h) \ominus \tilde{f}(t_0)}{h} = \dot{\tilde{f}}(t_0). \quad (15)$$

III. For all  $h > 0$ , sufficiently small  $\exists \tilde{f}(t_0) \ominus \tilde{f}(t_0 + h), \tilde{f}(t_0 - h) \ominus \tilde{f}(t_0)$  exist and the limits,

$$\lim_{h \rightarrow 0^+} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 + h)}{-h} = \lim_{h \rightarrow 0^-} \frac{\tilde{f}(t_0 - h) \ominus \tilde{f}(t_0)}{-h} = \dot{\tilde{f}}(t_0). \quad (16)$$

IV. For all  $h > 0$ , sufficiently small  $\exists \tilde{f}(t_0 + h) \ominus \tilde{f}(t_0), \tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)$  exist and the limits,

$$\lim_{h \rightarrow 0^+} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 + h)}{-h} = \lim_{h \rightarrow 0^+} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)}{-h} = \dot{\tilde{f}}(t_0). \quad (17)$$

For this differentiability, one needs to take different cases to obtain a fuzzy solution.

*Generalized Hukuhara Difference:* The thought of gH-weak subdifferential for Interval Valued Functions (IVFs) is introduced by generalised Hukuhara difference, which also offers a process for calculating the gh-weak subgradient.

In order to define generalised Hukuhara Differences [86], we take E to be the space of a convex nonempty set of X. Let  $\tilde{u}, \tilde{v}, \tilde{w} \in E$  then the generalized difference (gH) of  $\tilde{u}$  and  $\tilde{v}$ ,

$$\tilde{u} \ominus_g \tilde{v} \Leftrightarrow \begin{cases} \tilde{u} = \tilde{v} \oplus \tilde{w} \\ \text{or, } \tilde{v} = \tilde{u} \oplus (-1)\tilde{w}. \end{cases} \quad (18)$$

The generalised Hukuhara difference is given in other forms as follows,  ${}^\alpha \tilde{u} = [\underline{u}, \bar{u}]$  and  ${}^\alpha \tilde{v} = [\underline{v}, \bar{v}]$  is

$${}^\alpha \tilde{u} \ominus_g {}^\alpha \tilde{v} = [\min(\underline{u} - \underline{v}, \bar{u} - \bar{v}), \max(\underline{u} - \underline{v}, \bar{u} - \bar{v})]. \quad (19)$$

Based on the difference above, the Generalized Hukuhara derivative is given below.

Let fuzzy mapping  $\tilde{f} : I \rightarrow E$  and  $t_0 \in I$  then  $\tilde{f}$  is said to be generalized Hukuhara differentiable at  $t_0 \in I, \exists$  an element  $\dot{\tilde{f}}(t_0) \in E$  and given as,

$$\lim_{h \rightarrow 0^+} \frac{\tilde{f}(t_0 + h) \ominus \tilde{f}(t_0)}{h} = \dot{\tilde{f}}(t_0). \quad (20)$$

Now, with the help of examples, we calculate the Generalized Hukuhara difference between two fuzzy numbers.

**Example 4. When length of interval  ${}^\alpha \tilde{u} >$  length of interval  ${}^\alpha \tilde{v}$ .**

**Solution:**

At support, Let,  ${}^0 \tilde{u} = [5, 7], {}^0 \tilde{v} = [3, 4]$ ,

$$\begin{aligned} {}^0 \tilde{u} \ominus_g {}^0 \tilde{v} &= [\min(5 - 3, 7 - 4), \max(5 - 3, 7 - 4)] \\ &= [2, 3] \\ &= {}^0 \tilde{w}. \end{aligned} \quad (21)$$

So,

$$\begin{aligned} {}^0 \tilde{u} &= {}^0 \tilde{v} \oplus {}^0 \tilde{w} \\ &= [3, 4] + [2, 3] \\ &= [5, 7] \text{ (True).} \end{aligned} \quad (22)$$

**Example 5.** When length of interval  ${}^0\tilde{u} < \text{length of interval } {}^0\tilde{v}$ .

**Solution:**

Let  ${}^0\tilde{u} = [3, 4]$ ,  ${}^0\tilde{v} = [5, 7]$ ,

${}^0\tilde{u} \ominus_g {}^0\tilde{v} = [\min(3 - 5, 4 - 7), \max(3 - 5, 4 - 7)]$  and

${}^0\tilde{u} \ominus_g {}^0\tilde{v} = [-3, -2] = {}^0\tilde{w}$ . Then,

$$\begin{aligned} {}^0\tilde{u} &= {}^0\tilde{v} \oplus {}^0\tilde{w} \\ &= [5, 7] - [2, 3] \\ &= [3, 4] \text{ (True).} \end{aligned} \tag{23}$$

So, this  $gH$ -difference is always true. Now, we solve the same Example 3 by using  $gH$ -derivative,

$$\lim_{h \rightarrow 0^+} \frac{\tilde{f}(\tilde{x}_0 + h) \ominus_g \tilde{f}(\tilde{x}_0)}{h} = \lim_{h \rightarrow 0} \frac{[(\underline{x}_0 + h)^2, (\bar{x}_0 + h)^2] \ominus_g [\underline{x}_0^2, \bar{x}_0^2]}{h}. \tag{24}$$

After applying  $\alpha$ -cut, we have,

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{\tilde{f}(\tilde{x}_0 + h) \ominus_g \tilde{f}(\tilde{x}_0)}{h} &= \lim_{h \rightarrow 0} \frac{[(\underline{x}_0 + h, \bar{x}_0 + h)^2] \ominus_g [\underline{x}_0, \bar{x}_0]^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(\underline{x}_0 + h)^2, (\bar{x}_0 + h)^2] \ominus_g [\underline{x}_0^2, \bar{x}_0^2]}{h} \\ &= \lim_{h \rightarrow 0} \frac{[\min((\underline{x}_0 + h)^2 - \underline{x}_0^2, (\bar{x}_0 + h)^2 - \bar{x}_0^2), \max((\underline{x}_0 + h)^2 - \underline{x}_0^2, (\bar{x}_0 + h)^2 - \bar{x}_0^2)]}{h}. \end{aligned} \tag{25}$$

After solving, we have  $\lim_{h \rightarrow 0^+} \frac{\tilde{f}(\tilde{x}_0 + h) \ominus_g \tilde{f}(\tilde{x}_0)}{h} = [\min(2\underline{x}_0, 2\bar{x}_0), \max(2\underline{x}_0, 2\bar{x}_0)]$ . This expression is true for all fuzzy numbers, whether positive or negative.

**Modified Hukuhara Derivative:** The bounded and unique solution, known as the modified Hukuhara derivative (mH-derivative) provided as time increases, is its main benefit.

A function  $\tilde{f} : I \rightarrow E$  is Modified Hukuhara differentiable [87] if, at  $t_0 \in I$ ,  $\exists$  an element  $\dot{\tilde{f}}(t_0) \in E$  such that for all  $h > 0$  sufficiently small, there exists  $\tilde{f}(t_0 + h) \ominus \tilde{f}(t_0)$ ,  $\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)$  and the limits,

$$\lim_{h \rightarrow 0^+} \frac{\tilde{f}(t_0 + h) \ominus \tilde{f}(t_0)}{h} = \lim_{h \rightarrow 0^+} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)}{h} = \dot{\tilde{f}}(t_0). \tag{26}$$

The equivalent parametric representation is given as,

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{{}^0\tilde{f}(t_0 + h) \ominus {}^0\tilde{f}(t_0)}{h} &= \left[ \min \left\{ \lim_{h \rightarrow 0} \frac{\underline{f}(t_0 + h) - \underline{f}(t_0)}{h}, \lim_{h \rightarrow 0} \frac{\bar{f}(t_0 + h) - \bar{f}(t_0)}{h}, \lim_{h \rightarrow 0} \frac{\bar{f}(t_0 + h) - \underline{f}(t_0)}{h}, \right. \right. \\ &\quad \left. \left. \lim_{h \rightarrow 0} \frac{\bar{f}(t_0 + h) - \underline{f}(t_0)}{h} \right\}, \max \left\{ \lim_{h \rightarrow 0} \frac{\underline{f}(t_0 + h) - \underline{f}(t_0)}{h}, \right. \right. \\ &\quad \left. \left. \lim_{h \rightarrow 0} \frac{\underline{f}(t_0 + h) - \bar{f}(t_0)}{h}, \lim_{h \rightarrow 0} \frac{\bar{f}(t_0 + h) - \bar{f}(t_0)}{h}, \lim_{h \rightarrow 0} \frac{\bar{f}(t_0 + h) - \underline{f}(t_0)}{h} \right\} \right], \end{aligned} \tag{27}$$

and

$$\begin{aligned} \lim_{h \rightarrow 0^-} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)}{h} &= \left[ \min \left\{ \lim_{h \rightarrow 0} \frac{\underline{f}(t_0) - \underline{f}(t_0 - h)}{h}, \lim_{h \rightarrow 0} \frac{\bar{f}(t_0) - \bar{f}(t_0 - h)}{h}, \lim_{h \rightarrow 0} \frac{\bar{f}(t_0) - \bar{f}(t_0 - h)}{h}, \right. \right. \\ &\quad \left. \left. \lim_{h \rightarrow 0} \frac{\bar{f}(t_0) - \underline{f}(t_0 - h)}{h} \right\}, \max \left\{ \lim_{h \rightarrow 0} \frac{\underline{f}(t_0) - \underline{f}(t_0 - h)}{h}, \right. \right. \\ &\quad \left. \left. \lim_{h \rightarrow 0} \frac{\underline{f}(t_0) - \bar{f}(t_0 - h)}{h}, \lim_{h \rightarrow 0} \frac{\bar{f}(t_0) - \bar{f}(t_0 - h)}{h}, \lim_{h \rightarrow 0} \frac{\bar{f}(t_0) - \underline{f}(t_0 - h)}{h} \right\} \right]. \end{aligned} \tag{28}$$

Now, we solve Example 3 by using the Modified Hukuhara derivative. In which second and third terms in the above expression do not exist but the first and fourth always give similar solutions as the  $gH$ -Hukuhara derivative gives.

It is given as follows,  $\tilde{f}(x) = \tilde{x}^2$  at  $\tilde{x}_0$ .

$$\lim_{h \rightarrow 0^+} \frac{\alpha \tilde{f}(x_0 + h) \ominus \alpha \tilde{f}(x_0)}{h} = \left[ \min \left\{ \lim_{h \rightarrow 0} \frac{\underline{f}(x_0 + h) - \underline{f}(x_0)}{h}, \lim_{h \rightarrow 0} \frac{\underline{f}(x_0 + h) - \bar{f}(x_0)}{h}, \lim_{h \rightarrow 0} \frac{\bar{f}(x_0 + h) - \bar{f}(x_0)}{h}, \right. \right. \\ \left. \left. \lim_{h \rightarrow 0} \frac{\bar{f}(x_0 + h) - \underline{f}(x_0)}{h} \right\}, \max \left\{ \lim_{h \rightarrow 0} \frac{\underline{f}(x_0 + h) - \underline{f}(x_0)}{h}, \right. \right. \\ \left. \left. \lim_{h \rightarrow 0} \frac{\underline{f}(x_0 + h) - \bar{f}(x_0)}{h}, \lim_{h \rightarrow 0} \frac{\bar{f}(x_0 + h) - \bar{f}(x_0)}{h}, \lim_{h \rightarrow 0} \frac{\bar{f}(x_0 + h) - \underline{f}(x_0)}{h} \right\} \right]. \quad (29)$$

Then  $\lim_{h \rightarrow 0} \frac{(\underline{f}(x_0 + h) - \underline{f}(x_0))}{h}$  and  $\lim_{h \rightarrow 0} \frac{(\bar{f}(x_0 + h) - \bar{f}(x_0))}{h}$  gives  $2\tilde{x}_0$ ,  $2\bar{x}_0$ , respectively. But,  $\lim_{h \rightarrow 0} \frac{(\underline{f}(x_0 + h) - \bar{f}(x_0))}{h}$  and  $\lim_{h \rightarrow 0} \frac{(\bar{f}(x_0 + h) - \underline{f}(x_0))}{h}$  does not exist. So, to obtain a solution, we discard these two parts for all kinds of fuzzy values as mentioned in [87]. So, under the Modified Hukuhara derivative, the solution is  $[2\tilde{x}_0, 2\bar{x}_0]$ .

In the next section, we discuss the application of Fuzzy Differential equations.

*Some other types of Fuzzy Derivative:* Apart from the fuzzy derivative mentioned above, there are some other derivatives which are based on the definition of fuzzy derivative. They are: Dubois-Prade derivative [88], Goetschel-Voxman derivative [89], Seikkala derivative [90], Friedman-Ming-Kandel derivative [91], Some-order and reverse-order derivative (Yue and Guangyuan) (1998), SGH-derivative [92],  $\pi$ -derivative [93], g-derivative [94],  $H_2$ -derivative [95], and gr-derivative [96].

### 3|Classification of Fuzzy Differential Equations

#### Fuzzy Initial Value Problem (FIVP)

A differential equation with some initial conditions is called an initial value problem. A fuzzy differential equation with some fuzzy initial conditions is called a fuzzy initial value problem. Here, we consider the n-th order fuzzy initial value problem (FIVP) with the initial condition[97],

$$\begin{cases} \tilde{y}^n(s) = \tilde{f}(y, s), s_0 \leq s \leq S \\ \tilde{y}(s_0) = \tilde{y}_0. \end{cases} \quad (30)$$

In problem Equation (30),  $\tilde{y}$  is a fuzzy function of  $s$ ;  $\tilde{f}(y, s)$  be a fuzzy function of crisp interval  $s$  and  $\tilde{y}, \tilde{y}^n$  which are fuzzy variables, are the Hukuhara fuzzy derivatives of  $\tilde{y}$ . Now, we represent a second order linear fuzzy initial value problem(FIVP) with two initial conditions [98, 78], as an example,

$$\begin{cases} \tilde{y}''(s) = \tilde{f}'(y, s), 0 \leq s \leq 1 \\ \tilde{y}(0) = \tilde{\alpha} \\ \tilde{y}'(0) = \tilde{\beta}. \end{cases} \quad (31)$$

In problem Equation (31),  $\tilde{y}, \tilde{y}'$  are fuzzy functions of  $s$ ;  $\tilde{f}'(y, s)$  be a fuzzy function of crisp interval  $s$ ;  $\tilde{y}, \tilde{y}', \tilde{y}''$  which are fuzzy variables, are the Hukuhara fuzzy derivatives of  $\tilde{y}$  and  $\tilde{\alpha}, \tilde{\beta}$  denotes the symmetric fuzzy numbers [98].

We can solve the fuzzy initial value problem by using the method of Hukuhara derivative [97]. For details about the Fuzzy Initial Value Problem (FIVP), anyone can follow the papers [78, 99, 100, 101].

#### Fuzzy Boundary Value Problem (FBVP)

In this paper, we choose a fuzzy boundary value problem (FBVP) with fuzzy boundary values, i.e.,  $y(r) = \tilde{R}$ ,  $y(t) = \tilde{T}$ . Now, we consider a second order linear differential equation for a clear explanation.

We know the coefficients of the differential equation are not necessarily constant [102] but here we assume that these are constant. Now, We consider the n-th order fuzzy boundary value problem, then

$$\begin{cases} \frac{d^n y}{dx^n} + q_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + q_{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + q_1 \frac{dy}{dx} + q_0 y = f(x) \\ y(r) = \tilde{R} \\ y(t) = \tilde{T}. \end{cases} \quad (32)$$

In the boundary value problem in Equation (32), the coefficients  $q_{n-1}, q_{n-2}, \dots, q_1, q_0$  are fuzzy numbers and the function  $f'(x)$  is a fuzzy function.

Following this way, we consider, as an example, a 2nd order fuzzy boundary value problem with boundary values  $y(r) = \tilde{R}', y(t) = \tilde{T}'$ , then

$$\begin{cases} \frac{d^2 y}{dx^2} + q_1 \frac{dy}{dx} + q_2 y = f'(x) \\ y(r) = \tilde{R}' \\ y(t) = \tilde{T}'. \end{cases} \quad (33)$$

In the boundary value problem in Equation (33), the coefficients  $q_1$  and  $q_2$  are fuzzy numbers and the function  $f'(x)$  is a fuzzy function. For more expository information, we can follow [103].

We can solve FBVP by following methods [104], they are:

- (i) Matrix representation [102],
- (ii) Linear transformation [102]

For details about the Fuzzy Boundary Value Problem (FBVP), anyone can follow the papers [105, 106, 102, 107].

## Fuzzy Delay Differential Equation (FDDE)

A differential equation in which the state variable presents with a delayed argument is known as a delay differential equation (DDE) and a fuzzy differential equation where the state variable presents with a delayed argument is called a fuzzy delay differential equation. It is a fuzzy functional differential equation (functional DEs) that the derivatives of an unknown function at one point in time depend on the function's value at another point in prior time [108].

At first, we consider the n-th order Fuzzy Delay Differential Equation [109, 110],

$$D^n \tilde{u}(y) = \tilde{x}(y, \tilde{u}(y), \tilde{u}(y - \alpha)), \text{ for all fuzzy level sets } [0, 1]. \quad (34)$$

In Equation (34),  $\tilde{u}(y)$  denotes fuzzy function;  $\tilde{u}(y - \alpha)$  is denoted as fuzzy delay function. It can be solved by using the Hukuhara derivative and Zadeh extension principles [109].

For example, we show a first order fuzzy linear delay initial value differential equation [111],

$$\begin{cases} \tilde{y}'(s) = p_1 \tilde{y}(s) + p_2 \tilde{y}(s - s'), s = [s_0, S] \\ \tilde{y}(s) = \tilde{y}_0, s \in [s_0 - s', s_0]. \end{cases} \quad (35)$$

In the Equation (35),  $\tilde{y}(s)$ ,  $\tilde{y}(s - s')$ ,  $[s' < s]$  are fuzzy functions of  $s$ ;  $p_1, p_2, \tilde{y}_0$  are fuzzy numbers;  $s'$  denotes the delay function and  $\tilde{y}'(s)$  is a fuzzy derivative of  $\tilde{y}(s)$ .

From the paper [108], we can clearly understand some real-world fuzzy delay models which are constructed by fuzzy delay differential equations. These are:

- (i) Time-delay Malthusian growth model,
- (ii) Ehrlich ascites tumor model,
- (iii) Mackey-Glass time series model.

For details about the Fuzzy Delay Differential Equation (FDDE), anyone can follow the papers [108, 112, 109, 113].

## Fuzzy Fractional Differential Equation (FFDE)

Fractional differential equations solve ordinary differential equations to arbitrary order, which is non-integer. The process of way of ambiguity evolves over time is expressed through Fuzzy Fractional Differential Equation [114]. There are different types of fractional integral and fractional derivatives, which contain the fractional operator that helps solve the fuzzy fractional differential equation. They are

- (i) Fuzzy Riemann-Liouville Fractional Integral [97],
- (ii) Caputo-Type Fuzzy Fractional Derivatives [97],
- (iii) Fuzzy fractional Euler's method [115].

Here, we give two examples of Fuzzy Fractional Differential Equation with the above methods:

- (i) **Fuzzy Fractional Klein-Gordon Equation:**

$$\begin{cases} \frac{\partial^\beta \tilde{v}(y, r)}{\partial r^\beta} = \frac{\partial^2 \tilde{v}(y, r)}{\partial y^2} + p_1 \tilde{v}(y, r) + p_2 \tilde{v}^2(y, r) + p_3 \tilde{v}^3(y, r), & t \geq 0 \\ \tilde{v}(y, 0) = \tilde{v}_0, & y \in \mathbb{R}. \end{cases} \quad (36)$$

where,  $p_1, p_2, p_3$  are real constants and  $\beta$ , the parameter that explains the order of FFDE. We can solve Equation (36) using Caputo-Type Fuzzy Fractional Derivatives [97].

- (ii) **Fuzzy Fractional Initial Value Problem:**

$$\begin{cases} D^\beta \tilde{x}(z) = f(z, \tilde{x}) \\ \tilde{x}(0) = \tilde{x}_0, \beta \in (0, 1). \end{cases} \quad (37)$$

The Fuzzy Fractional Initial Value Problem Equation (37) is solved with the help of Fuzzy fractional Euler's method and Caputo-Type Fuzzy Fractional Derivatives using Modified Trapezoidal rule [114, 115].

For details about the Fuzzy Fractional Differential Equation (FFDE), anyone can follow the papers [116, 117, 118, 119].

## Fuzzy Partial Differential Equation (FPDE)

In mathematics, an ordinary differential equation is the basic idea of a fuzzy differential equation. A fuzzy partial differential equation contains an unknown fuzzy function of two or more variables and a fuzzy partial differential equation has the highest-order derivatives towards its order. Here, we present a general FPDE with partial differential operator, i.e.,

$$D(x(r)) = \tilde{F}(r, x(r), \tilde{\alpha}), \forall \tilde{\alpha} \in [0, 1] \quad (38)$$

Now, the different types of Fuzzy Partial Differential Equations (FPDE) are:

- (i) FPDE with certain boundary conditions,
- (ii) FPDE with initial conditions,
- (iii) Fuzzy Fractional Partial Differential Equations (FFPDE),
- (iv) Fuzzy Delay Partial Differential Equations (FDPDE).

In this portion, we consider a FPDE with certain boundary conditions, i.e.,

$$\begin{cases} \psi(D_x, D_r) \tilde{U}(x, r) = \tilde{F}(x, r, \tilde{\alpha}), & \forall \tilde{\alpha} \in [0, 1] \\ \tilde{U}(0, r) = c_1, \tilde{U}(x, 0) = c_2. \end{cases} \quad (39)$$

In the FPDE Equation (39),  $\psi(D_x, D_r)$  be the partial operator and polynomial with  $D_x, D_r$  which are constant coefficients;  $\tilde{F}(x, r, \tilde{\alpha})$  be the fuzzy function. For a detailed explanation of FPDE with certain boundary conditions using triangular fuzzy numbers, can follow the article [120].

Here, we mention some papers that described FPDE with initial conditions, FFPDE and FDPDE in an illustrative way, which gives a clear concept of fuzzy partial differential equations and various types of numerical methods to solve them easily. Those articles [121, 122] described FPDE with initial conditions; [123, 124] described FFPDE and [125] is helpful for FDPDE.

## 4|Solution of Fuzzy Differential Equations

There are several types of methods to find the solution of various Fuzzy Differential Equations [126, 127, 128]. They can be divided mainly into three stages, they are:

- (a) **Crisp conversion using different fuzzy derivative ,**
- (b) **Fuzzy mathematical transform using fuzzy derivative ,**
- (c) **Without Derivative Method,**
- (d) **Numerical Method.**

Now, we discuss the above mentioned processes in detail below,

(a) In the section of **Crisp conversion using different fuzzy derivative,**

- (i) **Method of Hukuhara Derivative(H-derivative Method):**

The general method for fuzzy value functions of an ordinary differential equation is the Hukuhara derivative method. In this approach, the first stated existence and uniqueness theorem to solve differential equations related to fuzzy value is named the Picard-Lindelof theorem. It can be used with two processes, i.e., the Hukuhara derivative and the Seikkala derivative. It is a general but lengthy process to solve FDE. For more details, we can confirm the paper [129, 130, 131].

- (ii) **Strongly Generalized Derivative Method(SGH-derivative Method):**

To reduce uncertainty, the area of the fuzzy differential equation can effortlessly be solved by the Strongly Generalized Derivative Method, which is related to the Hukuhara Derivative. This method is applicable to the fuzzy boundary value problem and fuzzy initial value problem. Initially, to solve a problem, a strongly generalized differentiability of type-1 named solution with increasing diameter and another is a strongly generalized differentiability of type-2 named solution with decreasing diameter [128]. Using this method, problems related to economics and engineering can be solved readily. To earn more knowledge with numerical examples, we can suggest the papers [131, 132, 133].

- (iii) **Generalized Derivative Method (gH-derivative Method):**

The generalized derivative method is the most popular way to solve fuzzy differential equations, especially various types of ordinary differential equations. GH-derivative Method is more facile process than H-derivative Method. To solve the fuzzy initial value problem and the initial value problem of interval-valued nonlinear systems of DE, basically, it processes with the existence and uniqueness to get the required solutions. For more details, we can follow the paper [129, 131, 134].

- (iv) **Granular Differentiability Method(gr-derivative Method):**

This is the proper rule based method whose solution can occupy the mid value of a crisp description approach and dependencies approach. The way of solving the initial value problem for an ordinary differential equation can be discussed as a granular generalization of Euler's method. During the rest of the problem solving, we can get the linguistic values of derivatives. We can figure out the papers to gain more knowledge [108, 135, 136].

- (v) **Fuzzy Differential Inclusions Method:**

The dynamic model is not narrated using the method of fuzzy differential inclusions. The important thing about this process is that the idea of the derivative of a fuzzy-number-valued function is absent. Some types of uncertain problems which come from the fuzzy partial differential equation can be solved by it. At the basic stage, to solve the fuzzy partial Differential equation, one FPDE has to be transformed to the membership degree of the leading equation from the fuzzy differential inclusions method and then solve the rest part numerically. Paper that describes this appropriately is [137, 131].

(b) In the section of **Fuzzy mathematical transform using fuzzy derivative**,

(i) **Fuzzy Sumudu Transform Method:**

To solve different types of engineering and applied physics related problems of fuzzy differential equations (FDEs) fluently, we can use the fuzzy Sumudu transform (FST) method. Actually, FDEs can be used to model uncertain nonlinear systems. At first, an algebraic equation is formed by FST instead of FDE and then is solved without helping a new frequency domain. This is an advanced method to solve the fuzzy differential equation with boundary value problems. For further details, for example, we can ensure the papers [138, 139].

(ii) **Laplace Transform Method:**

Laplace Transform is the prominent method used to solve the second order Fuzzy Partial Differential Equation. Solving the FPDE with Hukuhara differentiability of a function of the highest order that is defined by fuzzy improper integral to acquire the necessary solution of our main problem. Now, we describe the step-by-step solution, i.e., after choosing the differentiability's type, one can convert the differentiable functions into their parametric forms. Then, solve the DE problem with given initial and boundary conditions and find its length. And finally, come to the solution. We can suggest the paper [140, 141, 142].

(iii) **Mellin transform method:**

Mellin transform method is the most effective method to solve the fractional order differential equation. First of all, put the general solution in the place of the given problem using the left-sided Riemann–Liouville derivative and then find the final outcome. For more details, we can study [143].

(iv) **Fourier Sine transform Method:**

This method is used to simplify the wave equation with fuzzy initial conditions. The given problem can be solved with a generalized Hukuhara partial differential equation. The article [144] narrates the Fourier Sine transform Method with the example of a fuzzy initial value problem in an explicit way.

(c) In the section of **Without Derivative Method**,

(i) **Sup-norm Method:**

The fuzzy differential equation is also solved in a topological way. Firstly, the original problem is put in Banach space by two parametric forms of a given fuzzy differential equation and then the problem is solved numerically to simplify the FIVP using the Cauchy Method. So, the given method is also called the fuzzy Cauchy problem. To get more ideas and numerical details, we can follow the paper [145].

(ii) **Embedding method:**

This is a useful method for solving fuzzy ordinary differential equations (FODE). First of all, find the eigenvalue of the  $n * n$  matrix, which is formed of fuzzy number from the given non-homogeneous FODE. The fuzzy initial value problem (FIVP) can be solved easily by this method. Firstly, fuzzified the initial condition and then the sufficient condition of existence of a fuzzy solution is put in to process the remaining solution. For more clarity, we can study the papers [146, 147, 148].

(d) In the section of **Numerical Method**,

(i) **Variational iteration method (VIM):**

The variational iteration method (VIM) is used to solve first order linear fuzzy differential equations. VIM was introduced by He, J. H. [149], which is the modification of a general Lagrange multiplier. We solve the nonlinear fuzzy differential equation like this,

$$\tilde{L}[v(x)] = \tilde{N}[v(x)] = q(x). \quad (40)$$

In Equation (40),  $\tilde{L}$  be the linear operator;  $\tilde{N}$  denotes a nonlinear operator and  $q(x)$  is a continuous function. This type of problem is described in more detail and solved by step properly in paper [150].

(ii) **Adomian decomposition method (ADM):**

ADM is another popular method to solve first order linear fuzzy differential equations with generalized differentiability. This method defines an unknown function by an infinite series with a nonlinear operator. The paper [150] explains this method in detail with appropriate examples.

(iii) **Cubic Spline Function Approximation Method:**

This is a single-step method that helps to solve fuzzy initial value problems more quickly. It effectively solves FIVP and the suggested solution has too little complexity. After solving the Initial Value Problems using the Cubic Spline Function Approximation Method, we compare the results obtained second order Taylor's method. For further details with the numerical example, we can follow the paper [151].

(iv) **Runge-Kutta method of order three:**

Fuzzy initial value problems are also solved by the Runge-Kutta method of order three. Fuzzy differential equations with initial values are the general process to model dynamical systems under uncertainty. In the paper, [152], C. Duraisamy and B. Usha introduced this new process for getting the solution of FIVP. They explain this beautifully with clear examples in sections four and five of the above-mentioned paper. For more knowledge, one can follow the paper [153].

(v) **Trapezoidal Method:**

This is a simple numerical method to solve ordinary fuzzy differential equations. It helps to simplify many types of engineering and physical problems. This is a truly one step method of a higher convergence order. Trapezoidal Method is also acceptable for solving hybrid fuzzy differential equations. We can use the paper [154] to know more details with examples about this method.

(vi) **Taylor Method:**

It is the familiar method for solving fuzzy initial value problems where the unknown parameters to some plausible values are fixed. The fuzzy input is converted in some way into the fuzzy output defined by the corresponding crisp systems. This is also called a fuzzy input–fuzzy output (FIFO) system. We can study a perfect article about the Taylor Method for solving FIVP in [155, 156]. And, in Section 4 of this paper, they discussed fuzzy differential equations with numerical examples by the Taylor method of order  $p$  explicitly.

## 5| Applications of Fuzzy Differential Equations

In this section, we discussed different applications of the Fuzzy Differential Equations as follows:

(i) **Newton's law of cooling in fuzzy environment:**

Vasavi et al. [157] used the variation of constants formula under substantially generalised differentiability to interpret the principle on the application field, namely Newton's law of cooling. The model is fitted with several versions of first order linear fuzzy differential equations and derives solutions from many interpretations. We can also follow this particular application in the following related papers [158, 159, 160].

(ii) **Oil and Gas industry safety model in fuzzy environment:**

Mondal et al. [161] have worked on an oil and Gas industry safety model in a fuzzy environment. In this paper, they conceptualised a generalised triangular fuzzy number and its fundamental characteristics. A first-order linear system of the differential equation with fuzzy initial conditions and parameters, known as the fuzzy linear differential equation system, is developed in this paper. Finally, they used the classical technique to examine the problem from multiple angles and created a mathematical system model of the oil and gas industry using fuzzy numbers. The authors of the article [162] also designed a safety model for the mining industry under uncertainty and fuzziness with the help of differential equations.

(iii) **Vibrating spring mass system in fuzzy environment:**

Divya and Ganesan [104] have modelled a mechanical vibration system as a second-order ordinary differential equation given the mass, spring constant, damping, and external force. The string's initial displacement has been estimated and taken to be a fuzzy number due to measurement inaccuracies. This vibrating spring-mass system with fuzzy starting displacement has been solved using a fuzzy variation of the Sumudu transform approach. Lots of examples of vibration problems with the fuzzy environment were studied in the paper [163] by the scribes. The constructors of the paper [164] worked on this application.

(iv) **Fuzzy differential equations in different types of engineering and physical field:**

Shams et al. [165] suggested this method. It is illustrated with engineering examples, such as the Brine Tanks Problem, which also demonstrated how the series solution converged to the precise solution in closed form or in series. Ahmadian et al. [166] showed different approaches to fractional differential equation models in a fuzzy environment. These problems were found in engineering and physical fields (i.e., oscillations, resistor-capacitor (RC) circuits and viscoelasticity). We also notice that Sin et al. [167] experimented with the model of viscosity behaviour for non-Newtonian fluid and fuzzy fractional differential equation with variable coefficient using the tau method to find the solution of Kelvin-Voigt equation.

(v) **Mathematical model on Malaria disease in fuzzy environment:**

Singh et al. [168] have focused on seriously infected individuals of malaria transmission in both crisp and fuzzy contexts, and they have discussed a mathematical model. A criterion related to instances in which the illness has relapsed is taken into account. This study has examined the model's stability in clear and fuzzy contexts. Further, the creators of the papers [169, 170] were composed about this infliction by a fuzzy differential equation.

(vi) **Mathematical Model in tumor growth in fuzzy environment:**

Khaliq et al. [171] have developed a fuzzy environment to address a more precise mathematical tumour growth model to reduce the ambiguity of model parameters. Fuzzy differential equations in fuzzy mathematical models have represented the entire pattern of the tumour growth mechanism. The differential equation is changed into a system of two ordinary differential equations using the idea of the Generalised Hukuhara derivative. The article builder Rivaz et al. [172] constructed the integro-partial differential equations, which indicated the growth of a tumour characterised by the presence of cancer stem cells, which was the preliminary reason for treatment failure and tumour relapse.

(vii) **Mathematical model of diabetes in fuzzy environment:**

Mahata et al. [173] were the research investigations of this fuzzy mathematical model of diabetes. They observed that certain data cannot be known with precision in real-world situations. The idea of a system of linear differential equations with fuzzy initial conditions formed the basis of the diabetes model. The fuzzy model was used as the generalised Hukuhara derivative approach in the process of solving governing fuzzy differential equations. In the article, Mahata et al. [174] figured out the structure of glucose insulin regulation with diabetes mellitus under a fuzzy and crisp environment and the numerical solution of the fuzzy differential equation. Acharya et al. [175] formed a model of glucose distribution in the bloodstream via a neutrosophic differential equation. Authors of the research article [176] gave a clear concept of diabetes in a fuzzy environment.

(viii) **Diagonally Implicit Multistep Block Method for solving Fuzzy Differential Equation:**

Ramli and Majid [177] used the diagonally implicit multistep block approach of order four to approximate numerically the solution of fuzzy differential equations. It was commonly recognised that the multistep block approach was an accurate and efficient way to solve ordinary differential equations. They both structured another article [178] relating this method using predictor and corrector formula.

(ix) **Fuzzy differential equations in Control:**

Keivanimehr et al. [179] structured a fragmented fuzzy controller for Microgrids' optimal load shedding. Jahromi M.H.M. et al. [180] represented a hybrid process of maximum power point tracking of a network-connected photovoltaic system that is based on fuzzy logic controller and gravity search algorithm concept. Safari and Imani [181] presented the brain emotional learning based intelligent control problem that is tuned by a fuzzy inference system for the adaptive control of satellite attitude. The following articles [182, 183, 184, 185, 186, 187, 188, 189, 190], are also based on the fuzzy controller that is used in many engineering problems.

(x) **Application of Epidemic Modelling:**

Mahata et al. [191] solved the Susceptible-Infected-Susceptible (SIS) model, which was a type of epidemic model in fuzzy environments by various techniques. The fuzzy and interval environment was pondered as the imprecise parameter. The fuzzy differential inclusion method, fuzzy derivative method and extension principle method are the three ways to solve the given leading fuzzy differential equation, which also

helped to solve the interval differential equation. After solving the given DEs, all required numerical outcomes were consulted. Narayananmoorthy et al. [192] demonstrated predator-prey model in fuzzy environment using Differential Equation.

- (xi) **Application of Fuzzy differential equations in Earth's energy balance model and climate:** Pakdaman et al. [193] focused on using unsupervised kernel least mean square (UKLMS) to solve fuzzy dynamical differential equations (FDDEs). A kernel technique for an LMS adaptive filter was applied to determine how UKLMS, a nonlinear adaptive filter, operated. The multivariate function that was embedded to estimate UKLMS estimated the FDDE solution. Utilising the suggested algorithm, the Earth energy balance model was solved (EBM), which was initially regarded as a fuzzy differential equation.
- (xii) **Application of partial Fuzzy differential equations in computational Mechanics:** Pownuk [194] proposed an extremely effective solution algorithm. This algorithm was based on sensitivity analysis and the finite element method (or any other numerical method of solving PDE, such as FEM or BEM). One can tackle engineering challenges with thousands of degrees of freedom by using this approach. When modelling mechanical systems (structures) with ambiguous parameters, fuzzy partial differential equations might be used. We can repute these different types of research works, i.e., [97, 139], which are effective in the entire engineering and science field. Mondal and Roy [195] given a clear description of the Lagrange multiplier method with the solution of second order linear fuzzy ordinary differential equation in mechanics.
- (xiii) **Using Fuzzy differential equation in friction model:** Bede et al. [196] proposed a novel method of modelling friction through the use of fuzzy differential equations within the framework of the strongly generalised differentiability notion. A continuous fuzzification of the signum function was the crucial idea. Because a fuzzy differential equation's solutions were not unique, researchers can choose the one that most closely matches the behaviour of the real-world system we were modelling, allowing researchers to include expert knowledge in our model.
- (xiv) **Fuzzy transform with application to class of delay differential equations:** To solve a class of delay differential equations, a Picard-like numerical approach based on the F-transform was proposed by Tomasiello [197]. The suggested method used operational matrices and vectors with known quantities to approximate solutions in a non-recursive manner for linear instances. The suggested approach performed well numerically when compared to established solutions.
- (xv) **Application of Fuzzy differential equations in harvesting Model:** The quota harvesting model's stability analysis at equilibrium points in a fuzzy environment was examined by Paul et al. [198]. The authors represented three different types of fuzzy cases with equilibrium points and their feasibility in a fuzzy environment. And fuzzy differential equation processed the numerical simulations were used for the entire solution. Further, Paul et al. [199] worked on a fuzzy environment and formed the proportional harvesting model.
- (xvi) **Application of Fuzzy differential equation in solar collector:** The globe is now forced to search for new renewable energy sources due to the depletion of fossil fuels and fuel supplies. One such essential energy source that is used in many commercial and domestic operations is solar energy. This source of heat can be utilised to raise the air temperature used in blow drying procedures. Pandit et al. [200] produced a fuzzy nonlinear dynamical model with the mathematical model of such an event, which has imprecised parameters and/or initial conditions.
- (xvii) **Under granular differentiability, fuzzy delay differential equations and their applications:** Son et al.[108] proposed two models: time-delay growth and the Malthusian model. The second model, known as the Ehrlich ascites tumour model with delay, was a form of dynamical system employed in investigations involving vast volumes of tissues.
- (xviii) **Application of Stochastic Fuzzy differential equation:** Malinowski and Michta [201] discussed the existence and uniqueness of solutions to stochastic fuzzy differential equations driven by Brownian motion. Jafari, H. et al. [202] also studied fuzzy stochastic differential equations with fractional Brownian motion that can be applied to real world systems. The

relationship between beginning conditions and stability qualities was shown to be continuous. For instance, they used stochastic fuzzy differential equations in a population dynamics model.

(xix) **Fuzzy differential equations in heat conduction:**

Johansson et al. [203] considered a radially symmetric inverse heat conduction problem (IHCP) application of the fundamental solutions (MFS) approach. Here, Cauchy data was provided by the authors. An outer boundary and an inner fixed boundary were also mentioned for use to determine data in the radially symmetric IHCP, respectively. The authors also showed the MFS regularisation approach for the time-dependent heat equation. Hetmaniok et al. [204] considered an inverse heat conduction problem using the homotopy perturbation process. After reconstructing the functions, authors can describe the temperature and heat flux on the boundary. Khaleghi et al. [205] also solved the heat conduction concept under a fuzzy environment.

(xx) **Fuzzy concept in finger printing:**

As an application, the authors, Bede et al. [206], presented a novel way to fingerprint coding. Each fingerprint yielded a fuzzy-number-valued function. They used this function's Fourier transformations to conduct fingerprint coding. Using the above-mentioned finger codes, matching in huge fingerprint databases will be more efficient. We can gain more knowledge from the work of Zinoun [207].

(xxi) **Wave equation under fuzzy environment:**

The writers Almutairi et al. [208] applied two numerical techniques based on finite difference schemes, centred time centre space and implicit schemes to solve fuzzy wave equations. The purpose of this paper was to develop a new type of centred time centre space and implicit techniques for numerically solving fuzzy wave equations using the double parametric fuzzy number methodology.

(xxii) **HIV Modelling with fuzzy condition:**

The research work [209] explored a population model infected with the classical HIV virus in an imprecise environment. Two individual kinds of environments, i.e., interval and fuzzy environments, were used in this article, along with theoretical and numerical explanations. We can specify some papers to gain more knowledge [210, 211].

(xxiii) **COVID-19 Model in fuzzy environment:**

Melliani et al. [212] pointed out the prevalence of COVID-19 in Brazil. They modelled a Corona epidemic to show a relation with the stability of COVID-19 free and pandemic equilibrium points and the basic reproduction rate with fuzzy ambience. We also studied the research articles [213, 214] for more knowledge.

(xxiv) **Growth and Decay Model:**

This paper shows a first-order linear homogeneous ordinary differential equation with generalized triangular fuzzy numbers. Mondal et al. [215] used numerical illustration to find the perfect solution for this model. We also suggest the article [216] for another discussion.

(xxv) **Application of fuzzy differential equations of integer and fractional order in inventory management phenomena:**

The inventory model describes optimal lot-preserving strategies in logistics expertise. It may arise in the form of EOQ and EPQ models. Traditionally, differential equations of integer order manifest EOQ and EPQ models in existing literature concerning crisp decision phenomena. However, in reality, decision making procedures include imprecision due to a lack of clear knowledge about demand and price fluctuations in the market. Most of the inventory models under imprecision used fuzzy numbers, but they bypassed fuzzy calculus through different defuzzification techniques.

If a variable involved in decision making procedures goes through fuzzy liked vagueness, a fuzzy derivative based differential equation will be filled with a mathematical tool to describe the concerned phenomena. Using fuzzy Hukuhara and generalised Hukuhara derivatives, only few articles [217, 218, 219, 220, 221, 222, 223, 224] discussed inventory model in fuzzy differential equation approach. Furthermore, if the inventory model under fuzzy imprecision is also affected by experiences gained by interaction, memory will play an important role in this concern. Fuzzy fractional differential equation of Riemann-Liouville [225] and Caputo definition [226, 227, 228, 229, 230]

were used to illustrate inventory models under memory and uncertainty. The fuzzy differential equations are also used in Artificial neural network systems [231], Evolutionary algorithms [232] and Swarm Intelligence [233].

## 6|Prospects for Future Research in Fuzzy Differential Equations

Though fuzzy logic has been used in many soft computing based modern tools and gadgets of twenty first century. However, applications of fuzzy differential equations are constrained in a limited area despite of strong motivation for implementation of the theory to deal with situations of uncertainty associated with dynamical systems. Since the innovation of novel fuzzy theory, the fuzzy differential has been discussed in many theoretical attempts. Initially, people started working only on fuzzy initial conditions under the Hukuhara derivative. However, the approach faced drawbacks because the solution became a non-fuzzy Generalized Hukuhara and the Modified Hukuhara derivative came to remove the lacuna in the preceding approach. However, most of the theoretical and applied based work using the fuzzy differential equation was executed for many linear models. With the existing literature, it may not proceed to be an effective and efficient tool for describing non-linear models. If the challenges regarding the direction are fruitful, then a lot of uncertainty controlled dynamical models in engineering, biomathematics and operation research may be reconstructed with appropriate meanings and motivations.

In the current decade, fuzzy concepts are used in fractional calculus. Most of the discussions used fractional derivatives due to Riemann-Liouville and Caputo. Other contemporary definitions of functional derivatives, like Caputo-Fabrizio and Atangana-Baleany, may explore associating fuzzy fractional differential equations in future. Furthermore, economic problems and electronic models that incur memory and non-probabilistic uncertainty can be addressed using different fuzzy fractional differential equation approaches in the future.

People also work on complete fuzzy techniques so that fuzzy differential equations give the fuzzy models without converting them into their counter crisp part. This approach may be an effective consequence of the existing theory for the future.

## 7|Conclusions

The theory of differential equations has significantly large and divergent domains of applications as it addresses the dynamic nature of physical processes from mathematical perspectives. However, such mathematical modelling may carry a sense of imprecision in several layers and measures. Fuzzy set and logic are two of the contemporary philosophies that deal with uncertainty using mathematical structures and symbols. However, fuzzy valued calculus differs from its crisp analogue as fuzzy arithmetic differs from its crisp counterpart. Thus, the study of fuzzy differential equations emerges as essential and significant. In this paper, we have manifested almost all possible aspects and approaches of fuzzy differential equations. We have accumulated summaries of the works done to date using fuzzy differential equations in numerous theoretical and applied insights. Besides the detailed presentation of the research work that has been done, this article also hints at the challenges and scopes of the future. For enthusiastic academicians employing fuzzy differential equation theory, this article will be a strong manifesto with a detailed literature survey in the true sense.

From the detailed discussion in this paper, the following points come regarding research scopes and challenges in future: First, when a non-linear differential equation is considered under fuzzy uncertainty, the analytical discussion becomes complicated and almost has no clue to solve within fuzzy context. However, most of the mathematical models derived from physical scenarios are of non-linear types and also involve uncertainty. So, there are a lot of scopes and challenges to addressing the mentioned shortcomings in the existing literature. Second, the accumulation of memory, learning, forgetting and other human-related issues in the context of fuzzy differentials may advance the theory of fuzzy differential equations. However, the involvement of such advanced characteristics will be a challenging task for the future.

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## Author Contribution

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## Data Availability

All data supporting the reported findings in this research paper are provided within the manuscript.

## Conflicts of Interest

The authors declare that they have no known conflicts of interest or personal relationships that could have appeared to influence the work reported in this paper. There are no conflicts of interest between authors.

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